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Chapter 07

INDEFINITE INTEGRAL

Integration By Parts:

If u and v be two functions of x, then integral of the product of these two functions is given by

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

In applying the above rule care has to be taken in the selection of the first function (u) and the second function (v). Normally we use the following methods:

1. If in the product of the two functions, one of the function is not directly integrable (e.g. $\ln x, \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ etc) then we take it as the first function and the remaining function is taken as the second function. e.g. In the integration of $\int x\tan^{-1} x\,dx$, $\tan^{-1} x$ is taken as the first function and x as the second function.

2. If there is no other function, then unity is taken as the second function.e.g. In the integration of $\int \tan^{-1}x\,dx$, $\tan^{-1} x$ is taken as the first function and 1 as the second function.

3. If both of the functions are directly integrable, then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the first function.

(Inverse trigonometric, Logarithmic, Algebraic, Trigonometric, Exponential).

In the above as ILATE e.g. In the integration of $\int x \sin x dx, x$ is taken as the first function and $\sin x$ is taken as the second function.

An important Result:

$$\int e^x \left(f(x) \right) + f'(x)\, dx = e^x f(x) + c$$

Generalized Form:

Let u be a polynomial and v be a function which can be integrated repeatedly.

$$\int u v\, dx = uv^{(1)} - u'v^{(2)} + u''v^{(3)} - u'''v^{(4)} + ...$$

Where $u' = \frac{du}{dx}, u'' = \frac{du'}{dx}$ etc and $v^{(1)} = \int v\, dx, v^{(2)} = \int v^{(1)}\, dx$ etc..

Some Special Integrals:

- $\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{r} \cos(bx+c-\theta)$
- $\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{r} \sin(bx+c-\theta)$
- $\int xe^{ax} \cos(bx+c)\, dx = \frac{xe^{ax}}{r} \cos(bx+c-\theta) - \frac{e^{ax}}{r^2} \cos(bx+c-2\theta)$
- $\int xe^{ax} \sin(bx+c)\, dx = \frac{xe^{ax}}{r} \sin(bx+c-\theta) - \frac{e^{ax}}{r^2} \sin(bx+c-2\theta)$

  Where $a = r\cos\theta, b = r\sin\theta, r = \sqrt{a^2+b^2}$
- $\int a^x \sin(bx+c)\, dx = \frac{a^x}{(\log a)^2 + b^2} \left[\log a \sin(bx+c) - b\cos(bx+c) \right] + k$

❖ $\int a^x \cos(bx+c)\,dx = \frac{a^x}{(\log a)^2 + b^2}\left[\log a \cos(bx+c) + b\sin(bx+c)\right] + k$

❖ $\int x e^{ax} \sin(bx+c)\,dx = \frac{xe^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$

$-\frac{e^{ax}}{(a^2+b^2)^2}\left[(a^2-b^2)\sin bx - 2ab\cos bx\right] + k$

❖ $\int x e^{ax} \cos(bx+c)\,dx = \frac{xe^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$

$-\frac{e^{ax}}{(a^2+b^2)^2}\left[(a^2-b^2)\cos(bx+c) + 2ab\sin(bx+c)\right] + k$

Example-1:

$\int \log x\, dx$

Solution:

Take $f(x) = \log x,\ g(x) = 1$

$I = \int (\log x)(1)\, dx$

$= \log x \int 1\, dx - \int \left[(\log x)^1 \int 1\, dx\right] dx$

$= x\log x - \int \left(\frac{1}{x} \times x\right) dx$

$= x\log x - \int 1\, dx$

$= x\log x - x + c$

$= x(\log x - 1) + c.$

Example-2:

Evaluate $\int \sin^{-1} x\, dx$, $x \in (-1,1)$

Solution:

Take $f(x) = \sin^{-1} x$, $g(x) = 1$

$$\int \sin^{-1} x\, dx = \sin^{-1} x \int 1\, dx - \int \left[\left(\sin^{-1} x \right)^1 \int 1\, dx \right] dx$$

$$= x \sin^{-1} x - \int \left(\frac{1}{\sqrt{1-x^2}} . x\, dx \right)$$

$$= x \sin^{-1} x - \int \left(\frac{x}{\sqrt{1-x^2}}\, dx \right)$$

$$= x \sin^{-1} x + \frac{1}{2} \int \left(\frac{-2x}{\sqrt{1-x^2}}\, dx \right) \quad \left[\int \frac{f^1(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + C \right]$$

$$= x \sin^{-1} x + \frac{1}{\not 2} . \not 2 \sqrt{1-x^2} + C$$

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

Example-3:

Evaluate $\int \frac{x\, e^x}{(x+1)^2}\, dx$; $x \neq -1$.

Solution:

$$\int \frac{x\, e^x}{(x+1)^2}\, dx = \int \frac{(x+1-1)e^x}{(x+1)^2}\, dx$$

$$= \int \left[\frac{\cancel{(x+1)} e^x}{\cancel{(x+1)^2}} - \frac{e^x}{(x+1)^2} \right] dx$$

$$= \int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2} \right] dx$$

$$\left[\int e^x \left[f(x) + f^1(x) \right] dx = e^x\ f(x) + C \right]$$

$$= e^x \cdot \left(\frac{1}{x+1} \right) + C.$$