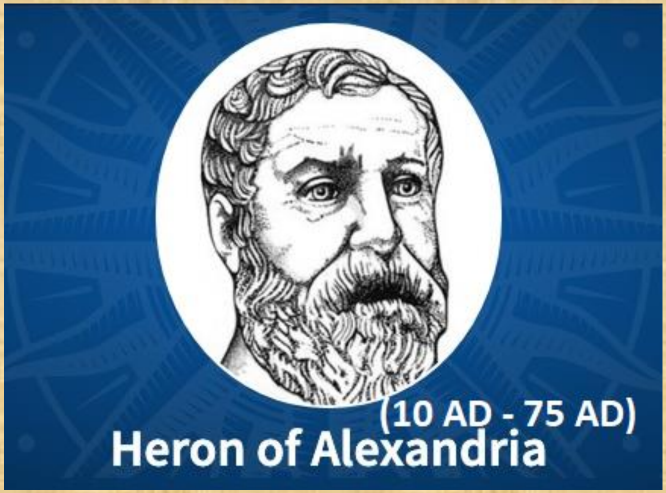
Chapter 12

HERON'S FORMULA

SCIENTIST STORY:

Heron (10 AD – 75 AD): Heron was born in about 10 AD possibly in Alexandria in Egypt. He worked in applied mathematics. His works on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book ideals with the area of squares, rectangles, triangles, trapezoids(trapezia), various other specialized quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc., Heron has derived the famous formula for the area of a triangle in terms of its three sides.



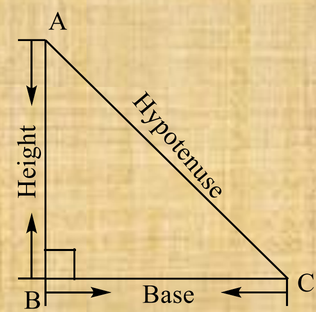
INTRODUCTION:

- In earlier classes, we have studied about various shapes and their properties.
- Also, we studied about the area and perimeter of these shapes.
- In this chapter, we will study the new formula for finding the area of a triangle, if the length of its three sides are given.

Basic Formulas:

➤ In Right Angled Triangle:

Area of triangle $ABC = \frac{1}{2} \times \text{base} \times \text{height}$



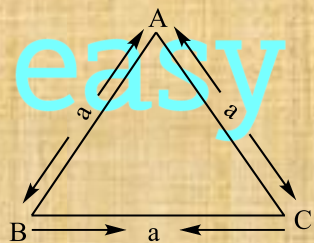
Where AB = height

BC = Base

AC = Hypotenuse

➤ In equilateral Triangle:

Area of an equilateral triangle $= \frac{\sqrt{3}}{4} a^2$

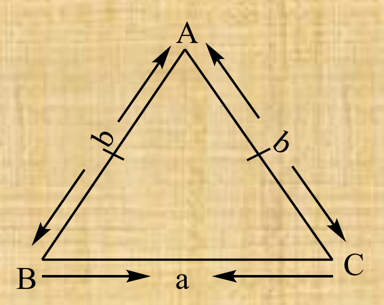


Where $AB = BC = CA = a$

a - side of the equilateral triangle.

➤ In Isosceles Triangle:

Area of an Isosceles triangle $= \frac{1}{4} \times a \times \sqrt{4b^2 - a^2}$



Where BC = a = Base

AB = AC = b = equal sides of length.

NOTE: Area of an isosceles Right–angled triangle, whose each equal side is, a is $\frac{a^2}{2}$.

SYNOPSIS:

Heron's Formula:

The formula given by heron about the area of a triangle is known as "Heron's" formula.

i.e., Area of a triangle $=\sqrt{S(S-a)(S-b)(S-c)}$

Where a, b, c are the sides of the triangle and

S = semi perimeter $=\dfrac{a+b+c}{2}$ = half of the perimeter of triangle.

NOTE:

- This formula is applicable to all types of triangles.
- This formula is helpful where it is not possible to find the height of the triangle easily.

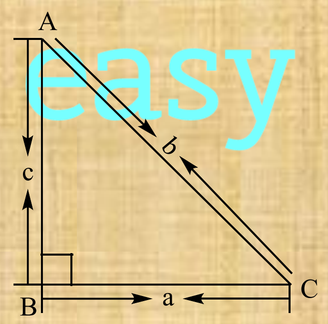
Semi perimeter of Different Triangles:

(1) In Right Angled Triangle:

Sides $BC=a$

$CA=b$

$AB=c$



Perimeter = a + b + c

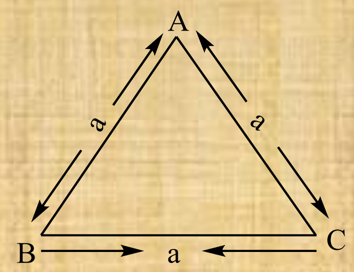
Semi perimeter $=\dfrac{a+b+c}{2}$

(2) In Equilateral Triangle:

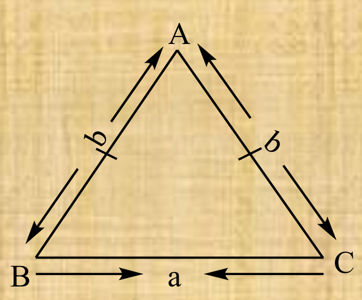
Sides BC = CA = AB = a

Perimeter = a + a + a = 3a

Semi perimeter $=\dfrac{a+a+a}{2}=\dfrac{3a}{2}$



(3) In Isosceles Triangle:



Sides: BC = a

CA = b = AB

Perimeter = a + b + b = a + 2b

Semi perimeter $= \dfrac{a+b+b}{2} = \dfrac{a+2b}{2}$

NOTE:

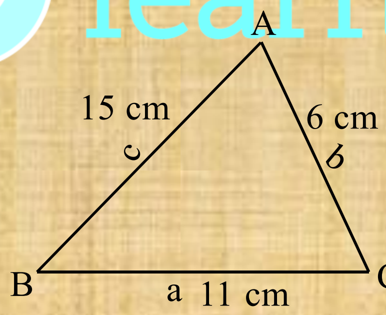
- Length of the longest altitude is the perpendicular distance from the opposite vertex to the smallest side of a triangle.
- Length of the smallest altitude is the perpendicular distance from the opposite vertex to the longest side of triangle.

Example: Find the Area of triangle whose sides are 11 cm, 6 cm, and 15 cm.

Sol: Here $BC = a = 11\,\text{cm}$

$CA = b = 6\,\text{cm}$

$AB = c = 15\,\text{cm}$



$$S = \frac{a+b+c}{2} = \frac{11+6+15}{2} = \frac{32}{2} = 16$$

Now,

$S - a = 16 - 11 = 5$

$S - b = 16 - 6 = 10$

$S - c = 16 - 15 = 1$

Area of Triangle $=\sqrt{S(S-a)(S-b)(S-c)}$

$$=\sqrt{16\times5\times10\times15}$$

$$=\sqrt{16\times5\times5\times2\times3\times5}$$

$$=4\times5\sqrt{30}$$

Area of Triangle $=20\sqrt{30}$ sq. c.m.

APPLICATION OF HERON'S FORMULA IN FINDING AREAS OF QUADRILATERALS:

To find the area of a quadrilateral using heron's formula, follow the steps.

Step-1:

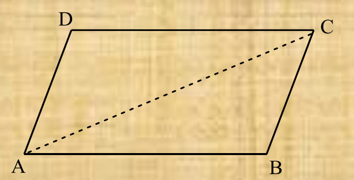
Divide the quadrilateral in to two triangular parts by joining any two of its opposite vertices.

Step-2:

Calculate the area of two triangles by using heron's formula.

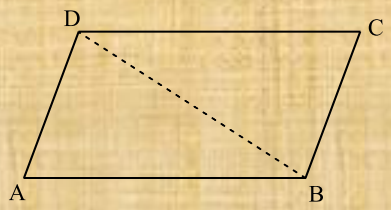
Step-3:

Area of quadrilateral ABCD = Area of ΔABC + Area of ΔADC



(OR)

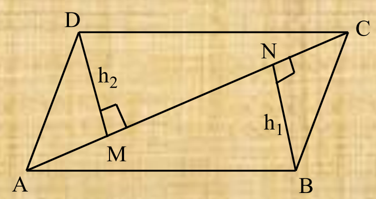
Area of quadrilateral ABCD = Area of ΔABD + Area of ΔBCD



NOTE:

If h_1 and h_2 are perpendicular distance of the diagonal AC of quadrilateral ABCD from vertices B and D respectively, then

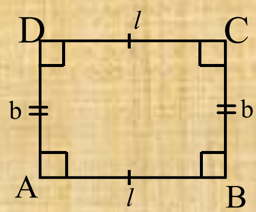
$$\text{Area} = \frac{1}{2} \times AC \times (h_1 + h_2).$$



Area of Some Special Quadrilaterals

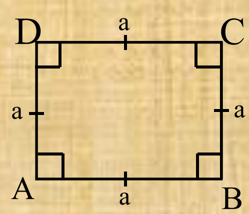
The following area formulae can be derived by Heron's Formula

Rectangle



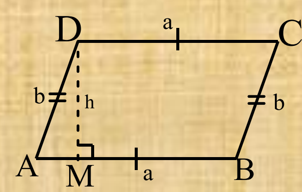
Let
l = length of Rectangle
b = breadth of Rectangle
(i) Perimeter = $2(l+b)$
(ii) Area = length x breadth
= l x b
(iii) Diagonal = $\sqrt{l^2+b^2}$

Square



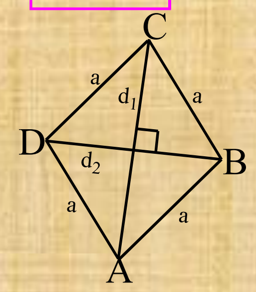
Let
a = side of a square
(i) Perimeter = 4a
(ii) Area = a^2
(iii) Diagonal = $\sqrt{a^2+b^2}$
$=\sqrt{2}a$

Parallelogram



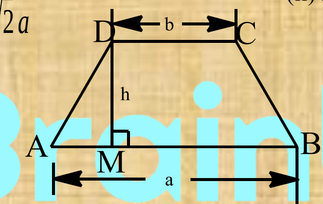
Let
Adjacent sides of parallelogram = a
Adjacent sides of parallelogram = b
height of parallelogram = h
(i) Perimeter = 2(a + b)
(ii) Area = Base x height
= a x h

Rhombus



Let
a = side of Rhombus
d_1= length of Diagonal AC
d_2= length of Diagonal BD
(i) Perimeter = 4a
(ii) Area = $\frac{1}{2}(d_1 \times d_2)$

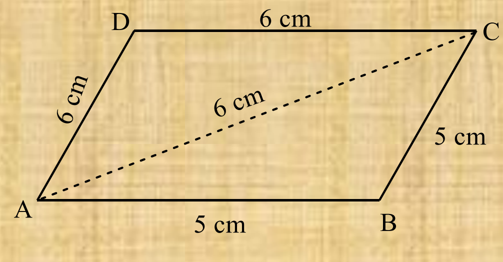
Trapezium



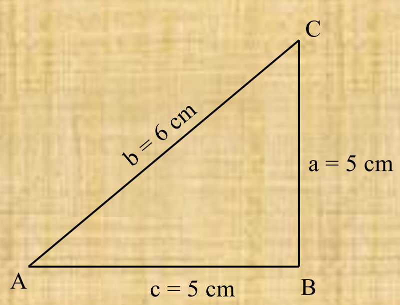
Let
a = one of length of parallel side
b = one of length of parallel side
h = distance between height of parallel lines
(i) Area = $\frac{1}{2}(a+b)\times h$

Ex: Find the area of the Quadrilateral ABCD, if AB = 5 cm, BC = 5 cm, CD = 6 cm, AD = 6cm and diagonal AC = 6 cm.

Sol: Diagonal AC divides the quadrilateral ABCD into two triangles ABC and ACD.



In ΔABC, $a=5$, $b=6$, $c=5$ cm



$$S=\frac{a+b+c}{2}=\frac{5+6+5}{2}=\frac{16}{2}=8$$

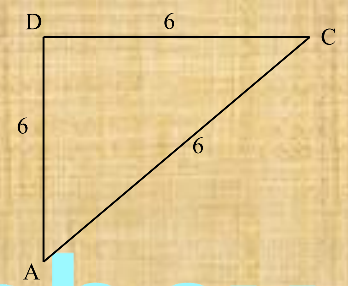
Area of $\Delta ABC=\sqrt{S(S-a)(S-b)(S-c)}$

$=\sqrt{8\times 3\times 2\times 3}$

$=\sqrt{16\times 9}$

$=12$

In ΔACD is an equilateral



Where $a=6$ cm

Area of $\Delta ACD=\frac{\sqrt{3}}{4}a^2=\frac{\sqrt{3}}{4}\times 6^2=\frac{\sqrt{3}}{\cancel{4}_{\cancel{2}}}\times \cancel{6}^3\times \cancel{6}^3=9\sqrt{3}\text{ cm}^2$.

$\therefore$ Area of Quadrilateral ABCD = (Area of ΔABC) + (Area of ΔACD)

Area of Quadrilateral ABCD $=\left(12+9\sqrt{3}\right)\text{cm}^2$.

SOLVED PROBLEMS

Ex-1: Find the area of triangle whose two sides are 18 cm and 10 cm and the perimeter is 42 cm.

Sol: Let the sides are $a=18$ cm , $b=10$ cm and $C=c$ cm

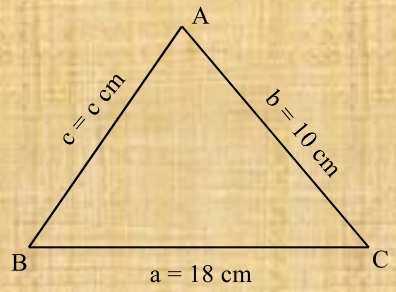
Q Perimeter of triangle = 42

$\Rightarrow a+b+c=42$

$\Rightarrow 18+10+c=42$

$\Rightarrow 28+c=42$

$\Rightarrow c=42-28=14\text{ cm}$



Now,

$$S=\frac{a+b+c}{2}=\frac{18+10+14}{2}=\frac{42}{2}=21$$

Area of triangle $=\sqrt{S(S-a)(S-b)(S-c)}$

Area of triangle $=\sqrt{21(21-18)(21-10)(21-14)}$

$=\sqrt{21(3)(11)(7)}$

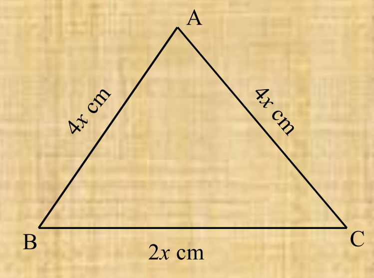
$=\sqrt{7\times3\times3\times11\times7}$

Area of triangle $=21\sqrt{11}\text{ cm}^2$.

Ex-2: The sides of a ΔABC are in the ratio 2: 4: 4 and its perimeter is 40 cm. Then, $ar(\Delta ABC)=K\sqrt{15}\text{ cm}^2$. Find the value of '$K$'?

Sol: Let the sides of ΔABC be $2x$, $4x$, $4x$

$\because$ Perimeter of triangle = 40



$\Rightarrow 2x + 4x + 4x = 40$

$\Rightarrow 10x = 40$

$\Rightarrow x = 4$ cm

So, the sides are

$2x = 2(4) = 8$ cm

$4x = 4(4) = 16$ cm

$4x = 4(4) = 16$ cm

$$S = \frac{a+b+c}{2} = \frac{8+16+16}{2} = \frac{40}{2} = 20$$

Area of Δ ABC $= \sqrt{S(S-a)(S-b)(S-c)}$

$= \sqrt{20(20-8)(20-16)(20-16)}$

$= \sqrt{(20)(12)(4)(4)}$

$= \sqrt{5 \times 4 \times 4 \times 3 \times 4 \times 4}$

Area of Δ ABC $= 16\sqrt{15}$

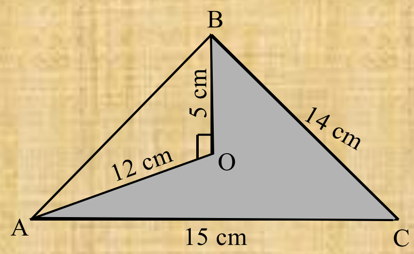
But,

$ar(\Delta\, ABC) = K\sqrt{15}$

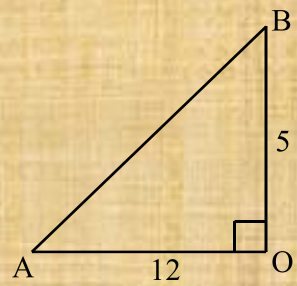
$\Rightarrow 16\sqrt{15} = K\sqrt{15}$

$\therefore K = 16$

Ex-3: Calculate the area of the shaded region.



Sol: Area of (Right angle) Δ AOB = $\frac{1}{2} \times \text{Base} \times \text{height}$



$$= \frac{1}{\cancel{2}} \times \cancel{12}^{6} \times 5$$

Area of Δ AOB $= 30\ \text{cm}^2$

Now, By Pythagoras Theorem,

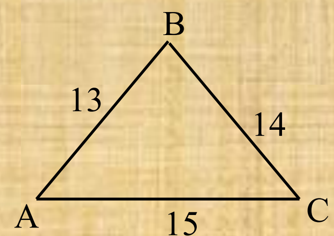
$$AB^2 = AO^2 + OB^2$$

$$= 12^2 + 5^2$$

$$AB^2 = 144 + 25 = 169$$

$$AB = \sqrt{169} = 13\ \text{cm}$$

Now Δ ABC $\Rightarrow$ a =14 cm , b =15 cm , c =13 cm



$$S = \frac{a+b+c}{2} = \frac{14+15+13}{2} = \frac{42}{2} = 21\ \text{cm}$$

Area of Δ ABC=$\sqrt{S(S-a)(S-b)(S-c)}$

Area of Δ ABC=$\sqrt{21(21-14)(21-15)(21-13)}$

$$= \sqrt{21(7)(6)(8)}$$

$$= \sqrt{7 \times 3 \times 7 \times 3 \times 2 \times 4 \times 2}$$

$$= 7 \times 3 \times 2 \times 2$$

$= 84 \text{ cm}^2$

Area of shaded region = (Area of ΔABC) – (Area of ΔAOB)

$= 84 - 30$

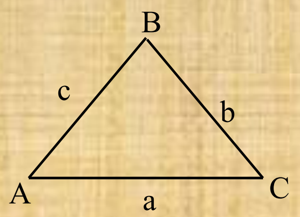
$\therefore$ Required Area $= 54 \text{ cm}^2$

Ex-4: Find the Area of the triangle. Length of whose sides are 32 cm, 40 cm and 24 cm. Hence, find the height corresponding to the smallest side.

Sol: Sides of the triangle are BC = a = 32 cm

CA = b = 40 cm

AB = c = 24 cm



Semi perimeter(S) $= \dfrac{a+b+c}{2} = \dfrac{32+40+24}{2} = \dfrac{96}{2} = 48 \text{ cm}$

Area of triangle $= \sqrt{S(S-a)(S-b)(S-c)}$

Area of triangle $= \sqrt{48(48-32)(48-40)(48-24)}$

$= \sqrt{48 \times 16 \times 8 \times 24}$

$= \sqrt{8 \times 6 \times 16 \times 8 \times 6 \times 4}$

$= 8 \times 6 \times 4 \times 2$

$= 384 \text{ cm}^2$

Now, the smallest side = 24 cm

Area of triangle $= \dfrac{1}{2} \times \text{Base} \times \text{height}$

$384 = \dfrac{1}{\cancel{2}} \times \cancel{24}^{12} \times \text{h}$

$$\Rightarrow h = \frac{\cancel{384}^{32}}{\cancel{12}_{1}}$$

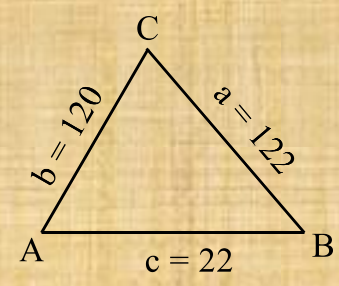
$\therefore$ Height of smallest side = 32 cm.

Ex-5: The triangular sidewalls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m. The advertisements yield an earning of ₹ 5000 per m^2 per year. A company hired one of its walls for 3 months. How much rent did it pay?

Sol: The sides of the triangular wall are $a = 122$ m

$b = 120$ m

$c = 22$ m



$$S = \frac{a+b+c}{2} = \frac{122+120+22}{2} = \frac{264}{2} = 132 \text{ m}$$

The area of a triangular sidewall is given by $= \sqrt{S(S-a)(S-b)(S-c)}$

Area $= \sqrt{132(132-122)(132-120)(132-22)}$

$= \sqrt{132(10)(12)(110)}$

$= \sqrt{12\times11\times10\times12\times11\times10}$

$= 12\times11\times10$

$= 1320 \text{ m}^2$

Rent for 1 year per m^2 = ₹ 5000

$\Rightarrow$ 12 months per m^2 = ₹ 5000

3 months per m^2 = ?

$\therefore$ For 3 months per $m^2 = \dfrac{5000\times3}{12}$

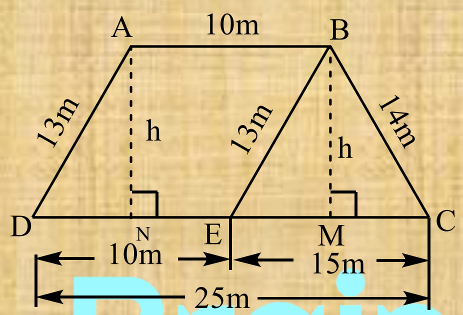
Rent for 3 months for $1320m^2 = \frac{5000\times 3}{\cancel{12}} \times \cancel{1320}^{110} =$ ₹ 16, 50,000

Ex-6: A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The non-parallel sides are 14 m and 13 m. find the area of the field.

Sol: Let the given field is in the form of a trapezium ABCD such that parallel sides are AB = 10 m and DC = 25 m.

Non-parallel sides are 13 m and 14 m we draw BE||AD. Such that BE = 13 m.

The given field is divided into two shapes.



(i) Parallelogram ABED

(ii) ΔBCE

(i) Area of ΔBCE

Sides of the triangle are a = 13 m , b = 14 m , c = 15 m

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21\,\text{m}$$

$$\text{Area of } \Delta BCE = \sqrt{S(S-a)(S-b)(S-c)}$$

$$\text{Area of } \Delta BCE = \sqrt{21(21-13)(21-14)(21-15)}.$$

$$= \sqrt{21(8)(7)(6)}$$

$$=\sqrt{7\times3\times4\times2\times7\times3\times2}$$

$$=7\times3\times2\times2$$

$$=84\text{ m}^2$$

Let the height of the ΔBCE corresponding to the side 15 m be h meters.

Area of a triangle $=\frac{1}{2}\times\text{base}\times\text{height}$

$$84=\frac{1}{2}\times15\times h$$

$$\Rightarrow\frac{\cancel{84}^{28}\times2}{\cancel{15}_{5}}=h$$

$$\therefore h=\frac{56}{5}\text{ m}$$

(ii) Area of parallelogram ABED = Base × height

$$=\cancel{10}^{2}\times\frac{56}{\cancel{5}}$$

Area of parallelogram ABED = 112 m^2

So, area of the field = area of $\Delta\, BCE$ + Area of parallelogram $ABED$

$$=84+112=196\text{ m}^2$$

$\therefore$ Required Area of field $=196\text{ m}^2$.

CONCEPT MAP

Heron's Formula

- **Heron's Formula**
  - Area of Triangle $= \sqrt{S(S-a)(S-b)(S-c)}$

    Where $S = \frac{a+b+c}{2}$ &

    a, b, c are sides of triangles
- **Application of heron's Formula in finding Areas of Quadrilaterals**
  - Rectangle
  - Square
  - Parallelogram
  - Rhombus
  - Trapezium